

# On The Interpolation of Injective or Projective Tensor Products of Banach Spaces

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**Abstract :** We prove a general result on the factorization of matrix-valued analytic functions. We deduce that if  $(E_0, E_1)$  and  $(F_0, F_1)$  are interpolation pairs with dense intersections, then under some conditions on the spaces  $E_0, E_1, F_0$  and  $F_1$ , we have

$$[E_0 \widehat{\otimes} F_0, E_1 \widehat{\otimes} F_1]_\theta = [E_0, E_1]_\theta \widehat{\otimes} [F_0, F_1]_\theta, \quad 0 < \theta < 1.$$

We find also conditions on the spaces  $E_0, E_1, F_0$  and  $F_1$ , so that the following holds

$$[E_0 \overset{\vee}{\otimes} F_0, E_1 \overset{\vee}{\otimes} F_1]_\theta = [E_0, E_1]_\theta \overset{\vee}{\otimes} [F_0, F_1]_\theta, \quad 0 < \theta < 1.$$

Some applications of these results are also considered.

## 1. Introduction, notation and background

All Banach spaces considered in this paper are complex. By an  $n$ -dimensional Banach space, we mean  $\mathbb{C}^n$  equipped with a norm.

If  $X$  and  $Y$  are Banach spaces, then  $\mathcal{L}(X, Y)$ ,  $X \overset{\vee}{\otimes} Y$  and  $X \widehat{\otimes} Y$  denote, respectively, the Banach space of bounded operators from  $X$  into  $Y$ , The closure of  $X \otimes Y$  in  $\mathcal{L}(X^*, Y)$  equipped with the induced norm, and the completion of  $X \otimes Y$  with respect to the projective tensor norm defined by :

$$\forall u \in X \otimes Y, \quad \|u\|_{\wedge} = \inf \left\{ \sum_{k=1}^m \|x_k\|_X \|y_k\|_Y : u = \sum_{k=1}^m x_k \otimes y_k \right\}$$

$X \overset{\vee}{\otimes} Y$  and  $X \widehat{\otimes} Y$  are called respectively the injective and projective tensor product of  $X$  and  $Y$ . In the case when  $X$  and  $Y$  are both finite-dimensional, we have

$$(X \widehat{\otimes} Y)^* = \mathcal{L}(X, Y^*) = X^* \overset{\vee}{\otimes} Y^*.$$

Using the preceding duality, we see that the results announced in the abstract are similar, in the finite-dimensional context.

Let  $E_0, E_1, F_0$  and  $F_1$  be finite-dimensional Banach spaces. The usual interpolation theorem asserts that

$$\| \cdot \|_{\mathcal{L}(E_\theta, F_\theta^*)} \leq \| \cdot \|_{[\mathcal{L}(E_0, F_0^*), \mathcal{L}(E_1, F_1^*)]_\theta},$$

where  $X_\theta$  denotes  $[X_0, X_1]_\theta$  for  $0 < \theta < 1$ .

The question we are interested in, is the following: Under what conditions on the spaces can one find a constant  $c$ , independent of the dimension of the considered spaces, so that

$$\| \cdot \|_{[\mathcal{L}(E_0, F_0^*), \mathcal{L}(E_1, F_1^*)]_\theta} \leq c \| \cdot \|_{\mathcal{L}(E_\theta, F_\theta^*)}.$$

We will see that the constant  $c$  can be majorized using the type 2 constants of the spaces  $E_0, E_1, F_0$  and  $F_1$  ( or the 2-convexity constants in the Banach lattice case).

We first recall some definitions and notation.  $\mathcal{E}_{n,m}$  denotes the space of complex  $n \times m$  matrices.  $A^*$  and  ${}^t A$  are, respectively, the adjoint and the transposed matrix of  $A$ . A matrix  $A \in \mathcal{E}_{n,m}$  will be identified with a linear operator from  $\mathbb{C}^m$  into  $\mathbb{C}^n$  using the canonical bases.

Let  $\delta$  be a norm on  $\mathcal{E}_{n,m}$ , the dual norm is defined by

$$\delta^*(A) = \sup \{ | \operatorname{tr} ({}^t B \cdot A) | : B \in \mathcal{E}_{n,m} \quad \text{and} \quad \delta(B) \leq 1 \}.$$

Let  $\mathcal{D}$ ,  $\overline{\mathcal{D}}$  and  $\partial\mathcal{D}$  denote, respectively the open unit disc in  $\mathbb{C}$ , its closure and its boundary. For  $z \in \mathcal{D}$  and  $t \in \partial\mathcal{D}$  we denote by  $P^z(t)$  the Poisson kernel:

$$P^z(t) = \frac{1 - |z|^2}{|t - z|^2}$$

and let  $dm$  be the Haar measure on  $\mathbb{T} \equiv \partial\mathcal{D}$ .

We use the notation  $A(\mathcal{D}, \mathcal{E}_{n,m})$  (resp.  $H^\infty(\mathcal{D}, \mathcal{E}_{n,m})$ ) to denote the set of analytic functions on  $\mathcal{D}$  valued in  $\mathcal{E}_{n,m}$  which are continuous on  $\overline{\mathcal{D}}$ , (resp. bounded on  $\mathcal{D}$ ).

We say that an operator  $u : X \rightarrow Y$  is  $p$ -summing for some  $p \geq 1$  if there is a constant  $c$  such that for all finite sequences  $x_1, \dots, x_n$  in  $X$  we have

$$\left( \sum_{k=1}^n \|u(x_k)\|_Y^p \right)^{1/p} \leq c \sup \left\{ \left( \sum_{k=1}^n |\xi(x_k)|^p \right)^{1/p} : \quad \xi \in X^*, \quad \|\xi\|_{X^*} \leq 1 \right\}.$$

We denote by  $\pi_p(u : X \rightarrow Y)$  the smallest constant  $c$  satisfying this property, and by  $\Pi_p(X, Y)$  the space of all  $p$ -summing operators from  $X$  into  $Y$ . This space equipped with the  $p$ -summing norm  $\pi_p$  is a Banach space.

We denote by  $\Gamma_2(X, Y)$  the set of all operators which factor through a Hilbert space. For an operator  $u \in \Gamma_2(X, Y)$ , we define the norm

$$\gamma_2(u : X \rightarrow Y) = \inf \{ \|A : X \rightarrow H\| \|B : H \rightarrow Y\| \}$$

where the infimum is taken over all factorizations of  $u$  of the form  $u = B.A$  and all Hilbert spaces  $H$ . The space  $\Gamma_2(X, Y)$  equipped with the preceding norm is a Banach space.

Let  $1 \leq p \leq +\infty$ . We denote by  $\ell_p^n$  the space  $\mathbb{C}^n$  equipped with the norm  $\|x\|_p = (\sum_1^n |x_k|^p)^{1/p}$ , and with the habitual change for  $p = +\infty$ .

Let  $\{g_n\}_{n \geq 1}$  be a sequence of independent identically distributed Gaussian real valued normal random variables on some probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . A Banach space  $X$  is called of type 2 (resp. Gaussian cotype 2), if there is a constant  $c$  such that for all  $x_1, \dots, x_n$  in  $X$  we have

$$\left\| \sum_{k=1}^n g_k x_k \right\|_{L^2(\Omega; X)} \leq c \left( \sum_{k=1}^n \|x_k\|^2 \right)^{\frac{1}{2}}$$

(resp.  $\geq c^{-1}(\sum \|x_k\|^2)^{1/2}$ ).

We denote by  $\tilde{T}_2(X)$  (resp.  $\tilde{C}_2(X)$ ) the smallest constant  $c$  for which this holds.

Let  $(e_1, \dots, e_n)$  be the canonical basis of  $\mathbb{C}^n$ . For any operator  $u : \ell_2^n \rightarrow X$  we define

$$\ell(u : \ell_2^n \rightarrow X) = \left\| \sum_{k=1}^n g_k u(e_k) \right\|_{L^2(\Omega; X)}.$$

For more details on operator ideals and the geometry of Banach spaces, we refer the reader to [P], [Pi1] and [LT1].

Concerning Banach lattices we refer the reader to [LT2]. We only recall the following definition: A Banach lattice  $X$  is called 2-convex (resp. 2-concave), if there exists a constant  $M$  so that for every choice of vectors  $x_1, \dots, x_n$  in  $X$

$$\left\| \left( \sum_{k=1}^n |x_k|^2 \right)^{1/2} \right\| \leq M \left( \sum_{k=1}^n \|x_k\|^2 \right)^{1/2}$$

(resp.  $\geq M^{-1}(\sum \|x_k\|^2)^{1/2}$ ). The smallest possible value of  $M$  is denoted  $M^{(2)}(X)$  (resp.  $M_{(2)}(X)$ ).

We assume the reader familiar with the complex interpolation method of Calderón, see for instance [C] or [BL].

We recall some definitions and results concerning the interpolation of families of finite-dimensional Banach spaces.

A family of norms  $\{\delta_\lambda\}_{\lambda \in \Lambda}$ , where  $\Lambda$  is any topological space, on  $\mathcal{E}_{n,m}$  is said to be *continuous* if for every  $A \in \mathcal{E}_{n,m}$  the function  $\lambda \mapsto \delta_\lambda(A)$  is continuous.

A continuous family of norms  $\{\delta_z\}_{z \in \overline{\mathcal{D}}}$  on  $\mathcal{E}_{n,m}$  is said to be *subharmonic* if for every  $\mathcal{E}_{n,m}$ -valued analytic function  $F$  defined on some domain  $\Omega \subset \mathcal{D}$  the function  $z \mapsto \text{Log } \delta_z(F(z))$  is subharmonic on  $\Omega$ .

If  $\{\delta_t\}_{t \in \partial\mathcal{D}}$  is a continuous family of norms on  $\mathcal{E}_{n,m}$ , there exists a unique continuous family of norms  $\{\delta_z\}_{z \in \overline{\mathcal{D}}}$  on  $\mathcal{E}_{n,m}$  which coincides with the original one on the boundary

and such that both  $\{\delta_z\}_{z \in \overline{\mathcal{D}}}$  and  $\{\delta_z^*\}_{z \in \overline{\mathcal{D}}}$  are subharmonic. This family ( or the family of spaces  $\{(\mathcal{E}_{n,m}, \delta_z)\}_{z \in \overline{\mathcal{D}}}$  ) is called the interpolation family with boundary data  $\{\delta_t\}_{t \in \partial\mathcal{D}}$ .

If  $\{\alpha_z\}_{z \in \overline{\mathcal{D}}}$  is an interpolation family of norms on  $\mathbb{C}^m$ ,  $\{\beta_z\}_{z \in \overline{\mathcal{D}}}$  is a subharmonic family of norms on  $\mathbb{C}^n$  and  $T \in A(\mathcal{D}, \mathcal{E}_{n,m})$  then the function

$$z \mapsto \text{Log} \|T(z) : (\mathbb{C}^m, \alpha_z) \rightarrow (\mathbb{C}^n, \beta_z)\|$$

is subharmonic.

Concerning the preceding two results and more details on the properties interpolation families we refer the reader to [CS],[CCRSW1] and [CCRSW2]. We also refer the reader to this last reference for the connection between this interpolation construction and the complex interpolation method of Calderón.

Let us finally describe the organization of this paper. In section 2, we prove some results which will be usefull for our problem, in particular, we prove a more general version of a theorem due to Gilles Pisier on the factorization of matrix-valued analytic functions [Pi2]. In section 3, we prove the announced results in the finite-dimensional case, using the theory of interpolation for families of Banach spaces. In section 4, we consider the infinite-dimensional case. Finally in section 5, we give some corollaries and applications of our results.

## 2. The factorization theorem

The following definition was introduced in an infinite-dimensional setting in [Pi2]. It reduces to the following in the matrix case :

**DEFINITION 2.1.** A norm  $\delta$  on  $\mathcal{E}_{n,m}$  is 2-convex, if for every  $A, B$  and  $C$  in  $\mathcal{E}_{n,m}$  the following holds

$$C^*C \leq B^*B + A^*A \quad \implies \quad \delta^2(C) \leq \delta^2(B) + \delta^2(A).$$

The next lemma is easy, and its proof is straightforward so we will omit it.

**Lemma 2.2.** Let  $\{(\alpha_t, \mathcal{E}_{m,m})\}_{t \in \Lambda}$  and  $\{(\beta_t, \mathcal{E}_{m,n})\}_{t \in \Lambda}$  be two continuous families, on  $\Lambda = \partial\mathcal{D}$  or  $\overline{\mathcal{D}}$ , formed of 2-convex norms on the corresponding matrix spaces. For  $t \in \Lambda$  and  $T \in \mathcal{E}_{n,m}$ , we define

$$\gamma_t(T) = \inf \{ \alpha_t(A) \cdot \beta_t(B) : \quad T = {}^t B \cdot A, \quad A \in \mathcal{E}_{m,m} \text{ and } B \in \mathcal{E}_{m,n} \}.$$

Then  $\{\gamma_t\}_{t \in \Lambda}$  is a continuous family of norms on  $\mathcal{E}_{n,m}$ .

In particular, the fact that  $\gamma_t$  is a norm, is a consequence of the hypothesis of 2-convexity made on the considered norms. We can now state and prove the main theorem of this section.

**Theorem 2.3.** Let  $\{(\alpha_z, \mathcal{E}_{m,m})\}_{z \in \overline{\mathcal{D}}}$  and  $\{(\beta_z, \mathcal{E}_{m,n})\}_{z \in \overline{\mathcal{D}}}$  be two subharmonic families of 2-convex norms. Then the family  $\{(\gamma_z, \mathcal{E}_{n,m})\}_{z \in \overline{\mathcal{D}}}$ , defined by

$$\gamma_z(T) = \inf \{ \alpha_z(A) \cdot \beta_z(B) : T = {}^t B \cdot A, A \in \mathcal{E}_{m,m} \text{ and } B \in \mathcal{E}_{m,n} \},$$

is subharmonic. More precisely, for every  $F \in A(\mathcal{D}, \mathcal{E}_{n,m})$  and every  $\varepsilon > 0$  there exist  $A \in H^\infty(\mathcal{D}, \mathcal{E}_{m,m})$  and  $B \in H^\infty(\mathcal{D}, \mathcal{E}_{m,n})$  such that for every  $z \in \mathcal{D}$  we have

$$F(z) = {}^t B(z) \cdot A(z)$$

and

$$\alpha_z(A(z)) \cdot \beta_z(B(z)) \leq (1 + \varepsilon) \exp \int_{\mathbb{T}} \text{Log } \gamma_t(F(t)) P^z(t) dm(t).$$

PROOF: It is clear that we only have to prove the last assertion of the theorem.

Let  $F \in A(\mathcal{D}, \mathcal{E}_{n,m})$  and  $\varepsilon > 0$ . There exist two continuous functions  $V : \partial\mathcal{D} \rightarrow \mathcal{E}_{m,m}$  and  $W : \partial\mathcal{D} \rightarrow \mathcal{E}_{m,n}$  such that for every  $t \in \partial\mathcal{D}$  we have

$$F(t) = {}^t W(t) \cdot V(t), \quad (2.1)$$

$$\alpha_t(V(t)) \leq 1 + \frac{\varepsilon}{2}, \quad (2.2)$$

$$\beta_t(W(t)) \leq \gamma_t(F(t)). \quad (2.3)$$

By the Wiener-Masani theorem [H, Lecture XI], there exists an outer function  $A \in H^\infty(\mathcal{D}, \mathcal{E}_{m,m})$  such that for almost every  $t \in \partial\mathcal{D}$

$$A^*(t)A(t) = V^*(t)V(t) + \frac{\varepsilon^2}{4s^2}I,$$

where  $I$  is the identity matrix and  $s = \sup\{\alpha_t(I) : t \in \partial\mathcal{D}\}$ .

Taking into account (2.2) and the 2-convexity of  $\alpha_t$  we obtain that for almost all  $t \in \partial\mathcal{D}$

$$\alpha_t(A(t)) \leq \left( \alpha_t^2(V(t)) + \frac{\varepsilon^2}{4} \right)^{1/2} \leq 1 + \varepsilon. \quad (2.4)$$

On the other hand, since  $V^*(t)V(t) \leq A^*(t)A(t) dm$  a.e. there exists a measurable function  $S : \partial\mathcal{D} \rightarrow \mathcal{E}_{m,m}$  such that

$$S^*(t)S(t) \leq I \quad \text{and} \quad V(t) = S(t)A(t) \quad dm \text{ a.e.} \quad (2.5)$$

We can now define  $B(z) = [{}^t A(z)]^{-1} \cdot {}^t F(z)$  for every  $z \in \mathcal{D}$ . Clearly  $B$  is an analytic function on  $\mathcal{D}$  which is also bounded since by (2.1), (2.5) and (2.3) we have

$$B(t) = [{}^t A(t)]^{-1} {}^t V(t) W(t) = {}^t S(t) W(t) \quad dm \text{ a.e.}$$

$$\beta_t(B(t)) \leq \beta_t(W(t)) \leq \gamma_t(F(t)) \quad dm \text{ a.e.} \quad (2.6)$$

By the subharmonicity of the norms and (2.4) , (2.6) we obtain that for every  $z \in \mathcal{D}$

$$\begin{aligned} F(z) &= {}^t B(z).A(z), \\ \alpha_z(A(z)) &\leq 1 + \varepsilon, \\ \beta_z(B(z)) &\leq \exp \int_{\mathbb{T}} \text{Log } \gamma_t(F(t)) P^z(t) dm(t). \end{aligned}$$

This achieves the proof of the theorem.

*Remark.* Note that if, in the preceding theorem, we take  $\alpha_z = \alpha$  ,  $\beta_z = \beta$  for all  $z \in \overline{\mathcal{D}}$ , we obtain the main part of a factorization theorem due to G. Pisier [Pi2].

**Corollary 2.4.** *Let  $\{(\alpha_z, \mathcal{E}_{m,m})\}_{z \in \overline{\mathcal{D}}}$  and  $\{(\beta_z, \mathcal{E}_{m,n})\}_{z \in \overline{\mathcal{D}}}$  be two interpolation families of 2-convex norms. Then the family  $\{(\gamma_z, \mathcal{E}_{n,m})\}_{z \in \overline{\mathcal{D}}}$ , defined by*

$$\gamma_z(T) = \inf\{\alpha_z(A).\beta_z(B) : T = {}^t B.A, A \in \mathcal{E}_{m,m} \text{ and } B \in \mathcal{E}_{m,n}\},$$

*is an interpolation family.*

PROOF: Using the preceding theorem we know that the family  $\{(\gamma_z, \mathcal{E}_{n,m})\}_{z \in \overline{\mathcal{D}}}$  is subharmonic.

Let  $\{N_z, \mathcal{E}_{n,m}\}_{z \in \overline{\mathcal{D}}}$  be a subharmonic family such that for every  $t \in \partial\mathcal{D}$  and for every  $T \in \mathcal{E}_{n,m}$  we have  $N_t(T) \leq \gamma_t(T)$ . Consider  $T \in \mathcal{E}_{n,m}$  and  $\omega \in \mathcal{D}$ , by definition, there exist  $A \in \mathcal{E}_{m,m}$  and  $B \in \mathcal{E}_{m,n}$ , such that

$$T = {}^t B.A \quad \text{and} \quad \alpha_{\omega}(A) = \beta_{\omega}(B) = \sqrt{\gamma_{\omega}(T)}.$$

Since  $\{\alpha_z\}_{z \in \overline{\mathcal{D}}}$  and  $\{\beta_z\}_{z \in \overline{\mathcal{D}}}$  are interpolation families, there exist analytic matrix-valued functions  $F \in H^{\infty}(\mathcal{D}, \mathcal{E}_{m,m})$  and  $G \in H^{\infty}(\mathcal{D}, \mathcal{E}_{m,n})$ , such that

$$F(\omega) = A, \quad G(\omega) = B \quad \text{and} \quad \alpha_t(F(t)) = \alpha_{\omega}(A), \quad \beta_t(G(t)) = \beta_{\omega}(B) \quad dm \text{ a.e.}$$

By the subharmonicity of the family  $\{N_z\}_{z \in \overline{\mathcal{D}}}$ , we obtain

$$\begin{aligned} N_{\omega}(T) &= N_{\omega}({}^t G(\omega).F(\omega)) \\ &\leq \int_{\partial\mathcal{D}} N_t({}^t G(t).F(t)) P^z(t) dm(t) \\ &\leq \int_{\partial\mathcal{D}} \gamma_t({}^t G(t).F(t)) P^z(t) dm(t) \\ &\leq \int_{\partial\mathcal{D}} \alpha_t(F(t)) \beta_t(G(t)) P^z(t) dm(t) \\ &= \gamma_{\omega}(T). \end{aligned}$$

We have proved that

$$\forall \omega \in \mathcal{D}, \quad \forall T \in \mathcal{E}_{n,m}, \quad \gamma_\omega(T) = \sup\{N_\omega(T)\}$$

where the supremum is taken over the set of all subharmonic families  $\{(N_z, \mathcal{E}_{n,m})\}_{z \in \overline{\mathcal{D}}}$  of norms satisfying the boundary condition:

$$\forall t \in \partial\mathcal{D}, \forall T \in \mathcal{E}_{n,m}, \quad N_t(T) \leq \gamma_t(T).$$

This proves that  $\{(\gamma_z, \mathcal{E}_{n,m})\}_{z \in \overline{\mathcal{D}}}$  is an interpolation family. (cf. [CS]).

The next result expresses the stability of the property of 2-convexity under the interpolation construction.

**Theorem 2.5.** *Let  $\{(\delta_t, \mathcal{E}_{n,m})\}_{t \in \partial\mathcal{D}}$  be a continuous family of 2-convex norms. Then the interpolation family  $\{(\alpha_z, \mathcal{E}_{n,m})\}_{z \in \overline{\mathcal{D}}}$  obtained from the preceding one consists of 2-convex norms.*

PROOF: For  $z \in \mathcal{D}$  and  $A \in \mathcal{E}_{n,m}$  define

$$\begin{aligned} \Delta_z(A) = \inf\{\lambda > 0 : \exists F \in H^\infty(\mathcal{D}, \mathcal{E}_{n,m}) \text{ with } A^*A \leq \lambda^2 F^*(z)F(z) \\ \text{and } \delta_t(F(t)) \leq 1 \text{ dm a.e.}\}. \end{aligned}$$

**Claim.** For every  $z$  in  $\mathcal{D}$  we have  $\Delta_z = \delta_z$ .

Indeed, if  $\delta_z(A) = 1$  then by Theorem I. from [CCRSW1] we can find an element  $F \in H^\infty(\mathcal{D}, \mathcal{E}_{n,m})$  such that  $F(z) = A$  and  $\delta_t(F(t)) \leq 1$  dm a.e, hence  $\Delta_z(A) \leq 1$ .

Conversely, if  $\Delta_z(A) < 1$  there exists an analytic function  $F \in H^\infty(\mathcal{D}, \mathcal{E}_{n,m})$  such that  $A^*A \leq F^*(z)F(z)$  and  $\delta_t(F(t)) \leq 1$  dm a.e. Thus there exists a contraction  $T$ , i.e.  $T^*T \leq I$ , such that  $A = TF(z)$ . For  $\omega \in \mathcal{D}$  let us consider  $G(\omega) = TF(\omega)$ . Clearly  $G \in H^\infty(\mathcal{D}, \mathcal{E}_{n,m})$ ,  $G(z) = A$  and for almost all  $t \in \partial\mathcal{D}$  we have  $G^*(t)G(t) \leq F^*(t)F(t)$ , so  $\delta_t(G(t)) \leq \delta_t(F(t)) \leq 1$ . This proves the claim.

Let us consider  $A, B$  and  $C$  in  $\mathcal{E}_{n,m}$  and  $\varepsilon > 0$ .

Define  $\alpha = \delta_z(A)$  and  $\beta = \delta_z(B)$ . Without loss of generality, we may assume that  $\alpha^2 + \beta^2 = 1$ . Then there exist  $F$  and  $G$  in  $H^\infty(\mathcal{D}, \mathcal{E}_{n,m})$  such that  $F(z) = A, G(z) = B$  and for almost all  $t \in \partial\mathcal{D}$   $\delta_t(F(t)) = \alpha, \delta_t(G(t)) = \beta$ .

By the Wiener-Masani theorem [H, Lecture XI], there exists an outer function  $L \in H^\infty(\mathcal{D}, \mathcal{E}_{n,m})$  such that

$$L^*(t)L(t) = F^*(t)F(t) + G^*(t)G(t) + \frac{\varepsilon^2}{s^2}I \quad dm \text{ a.e.} \quad (2.7)$$

with  $s = \sup\{\delta_t(I) : t \in \partial\mathcal{D}\}$ .

By (2.7) and using the 2-convexity of  $\delta_t$  we obtain

$$\delta_t(L(t)) \leq (\alpha^2 + \beta^2 + \varepsilon^2)^{1/2} \leq 1 + \varepsilon \quad dm \text{ a.e.} \quad (2.8)$$

Take  $x \in \mathbb{C}^m$  and  $h \in H^\infty(\mathcal{D}, \mathbb{C}^m)$  such that  $h(z) = x$ , and consider the function  $k \in H^\infty(\mathcal{D}, \mathbb{C}^{2n})$  defined by

$$k(\omega) = \begin{bmatrix} F(\omega)h(\omega) \\ G(\omega)h(\omega) \end{bmatrix}.$$

Since  $k$  is analytic, the function  $\omega \mapsto \log \|k(\omega)\|_2$  is subharmonic, (where  $\|\cdot\|_2$  is the Hilbertian norm), and we may write:

$$\left( \|Ax\|_2^2 + \|Bx\|_2^2 \right)^{1/2} = \|k(z)\|_2 \leq \exp \int_{\mathbb{T}} \log \|k(t)\|_2 P^z(t) dm(t). \quad (2.9)$$

By (2.7) ,(2.9) and the hypothesis we obtain

$$\|Cx\|_2 \leq \exp \int_{\mathbb{T}} \log \|L(t)h(t)\|_2 P^z(t) dm(t). \quad (2.10)$$

By the results of [CCRSW1] on the interpolation of Hilbertian families of norms, we know that the family  $\{y \mapsto \|L(\omega)y\|_2\}_{\omega \in \overline{\mathcal{D}}}$  is an interpolation family of norms on  $\mathbb{C}^m$ . Consequently, by taking the infimum in (2.10) over all  $h \in H^\infty(\mathcal{D}, \mathbb{C}^m)$  such that  $h(z) = x$  we obtain

$$\|Cx\|_2 \leq \|L(z)x\|_2$$

but  $x \in \mathbb{C}^m$  is arbitrary, so we have

$$C^*C \leq L^*(z)L(z). \quad (2.11)$$

Using (2.8) and (2.11) we obtain that  $\Delta_z(C) \leq 1 + \varepsilon$  and the result follows since  $\varepsilon$  is arbitrary.

### 3. The finite-dimensional case

This section is devoted to the proof of the following theorem.

**Theorem 3.1.** *Let  $\{E_z\}_{z \in \overline{\mathcal{D}}}$  and  $\{F_z\}_{z \in \overline{\mathcal{D}}}$  be two interpolation families of  $m$ -dimensional and  $n$ -dimensional spaces respectively. If  $T \in \mathcal{E}_{n,m}$  and  $z \in \overline{\mathcal{D}}$ , define  $\|T\|_z$  as the operator norm :  $\|T : E_z \rightarrow F_z^*\|$  and  $\{\|\cdot\|_{[z]}\}_{z \in \overline{\mathcal{D}}}$  as the interpolation family of norms on  $\mathcal{E}_{n,m}$  extending  $\{\|\cdot\|_t\}_{t \in \partial\mathcal{D}}$ . Then*

$$\forall z \in \overline{\mathcal{D}}, \forall T \in \mathcal{E}_{n,m} \quad \|T\|_z \leq \|T\|_{[z]} \leq c\|T\|_z,$$

with the following estimates for  $c$  :

1.  $c \leq \left( \sup \{ \tilde{T}_2(E_t) \cdot \tilde{T}_2(F_t) : t \in \partial\mathcal{D} \} \right)^2$ .
2.  $c \leq 2\sqrt{\frac{2}{\pi}} \left( \sup \{ \tilde{T}_2(E_t) \cdot M^{(2)}(F_t) : t \in \partial\mathcal{D} \} \right)^2$  if the canonical basis of  $\mathbb{C}^n$  is a 1-unconditional basis for  $F_t$  for every  $t \in \partial\mathcal{D}$ .
3.  $c \leq \frac{32}{\pi} \left( \sup \{ M^{(2)}(E_t) \cdot M^{(2)}(F_t) : t \in \partial\mathcal{D} \} \right)^{5/2}$  if the canonical basis of  $\mathbb{C}^m$  (resp.  $\mathbb{C}^n$ ) is a 1-unconditional basis for  $E_t$  (resp.  $F_t$ ) for every  $t \in \partial\mathcal{D}$ .

The proof of this theorem uses several lemmas which will be listed and proved in what follows.

**Lemma 3.2.** *Let  $\{E_z\}_{z \in \overline{\mathcal{D}}}$  be an interpolation family of  $m$ -dimensional spaces. If  $T \in \mathcal{E}_{n,m}$  and  $z \in \overline{\mathcal{D}}$ , define  $\pi_z^E(T) = \pi_2(T : E_z^* \rightarrow \ell_2^n)$ , and let  $\{\pi_{[z]}^E\}_{z \in \overline{\mathcal{D}}}$  be the interpolation family of norms on  $\mathcal{E}_{n,m}$  extending  $\{\pi_t^E\}_{t \in \partial\mathcal{D}}$ . Then*

$$\forall z \in \overline{\mathcal{D}}, \forall T \in \mathcal{E}_{n,m} \quad \pi_z^E(T) \leq W^E(z) \pi_{[z]}^E(T) \quad (*)$$

with  $W^E(z) = \exp \int_{\mathbb{T}} \text{Log } \tilde{T}_2(E_t) P^z(t) dm(t)$ .

PROOF of lemma 3.2. Note that if  $X$  is  $m$ -dimensional then for every  $T \in \mathcal{E}_{n,m}$  one has

$$\pi_2(T : X^* \rightarrow \ell_2^n) \leq \ell(t^* T : \ell_2^n \rightarrow X) \leq \tilde{T}_2(X) \cdot \pi_2(T : X^* \rightarrow \ell_2^n). \quad (3.1)$$

The first inequality follows trivially from the definition, and the second one is well-known and classical, see for instance [P,21.3.5].

Define  $\ell_z(T) = \ell(t^* T : \ell_2^n \rightarrow E_z)$  for  $z \in \overline{\mathcal{D}}$  and  $T \in \mathcal{E}_{n,m}$ . Clearly  $\{\ell_z\}_{z \in \overline{\mathcal{D}}}$  is a subharmonic family of norms on  $\mathcal{E}_{n,m}$ . Using the result of [CS] recalled in section 1. we obtain that the function

$$z \longmapsto \text{Log} \left\| I : (\mathcal{E}_{n,m}, \pi_{[z]}^E) \rightarrow (\mathcal{E}_{n,m}, \ell_z) \right\|$$

is subharmonic, so by (3.1) we may write

$$\forall z \in \overline{\mathcal{D}}, \quad \left\| I : (\mathcal{E}_{n,m}, \pi_{[z]}^E) \rightarrow (\mathcal{E}_{n,m}, \ell_z) \right\| \leq W^E(z).$$

A second use of (3.1) yields the result.

**Lemma 3.3.** *Let  $\{E_z\}_{z \in \overline{\mathcal{D}}}$  be an interpolation family of  $m$ -dimensional spaces, and assume that the canonical basis of  $\mathbb{C}^m$  is a 1-unconditional basis for  $E_t$  for every  $t \in \partial\mathcal{D}$ . If  $\{\pi_z^E\}_{z \in \overline{\mathcal{D}}}$  and  $\{\pi_{[z]}^E\}_{z \in \overline{\mathcal{D}}}$  are defined as in lemma 3.2 then*

$$\forall z \in \overline{\mathcal{D}}, \forall T \in \mathcal{E}_{n,m} \quad \pi_z^E(T) \leq W^E(z) \pi_{[z]}^E(T) \quad (*)$$

with  $W^E(z) = \frac{2}{\sqrt{\pi}} \exp \int_{\mathbb{T}} \text{Log } M^{(2)}(E_t) P^z(t) dm(t)$ .

PROOF of lemma 3.3. It is well-known ( see [Pi1,Chapter 8]) that if  $X$  is an  $m$ -dimensional complex Banach lattice, then for every  $T \in \mathcal{E}_{n,m}$  one has

$$\frac{\sqrt{\pi}}{2} \pi_2(T : X^* \rightarrow \ell_2^n) \leq \left\| \left( \sum_{k=1}^n |^t T(e_k)|^2 \right)^{1/2} \right\|_X \leq M^{(2)}(X) \pi_2(T : X^* \rightarrow \ell_2^n) \quad (3.2)$$

For  $T \in \mathcal{E}_{n,m}$  and  $z \in \overline{\mathcal{D}}$ , we define

$$\ell_z(T) = \left\| \left( \sum_{k=1}^n |^t T(e_k)|^2 \right)^{1/2} \right\|_{E_z} = \left\| \{^t T(e_k)\}_{1 \leq k \leq n} \right\|_{E_z(\ell_2^n)}$$

By the results of [He] on the interpolation of families of Banach lattices,  $E_z$  is an  $m$ -dimensional complex Banach lattice and  $\{E_z(\ell_2^n)\}_{z \in \overline{\mathcal{D}}}$  is an interpolation family. Consequently,  $\{\ell_z\}_{z \in \overline{\mathcal{D}}}$  is subharmonic, and the proof can be completed as in lemma 3.2 using (3.2) instead of (3.1).

*Remark.* Note that using the same notation as in lemma 3.2 and lemma 3.3, we also have

$$\pi_{[z]}^E(T) \leq \pi_z^E(T)$$

for all  $z \in \overline{\mathcal{D}}$  and all  $T \in \mathcal{E}_{n,m}$ . This will not be used in the sequel.

**Lemma 3.4.** *Let  $\{E_z\}_{z \in \overline{\mathcal{D}}}$  and  $\{F_z\}_{z \in \overline{\mathcal{D}}}$  be two interpolation families of  $m$ -dimensional and  $n$ -dimensional spaces respectively. Assume that both families satisfy the conclusion  $(*)$  of lemma 3.2 or lemma 3.3. For every  $z \in \overline{\mathcal{D}}$  and every  $T \in \mathcal{E}_{n,m}$  define  $\gamma_z(T) = \gamma_2(T : E_z \rightarrow F_z^*)$  and let  $\{\gamma_{[z]}\}_{z \in \overline{\mathcal{D}}}$  be the interpolation family of norms on  $\mathcal{E}_{n,m}$  extending  $\{\gamma_t\}_{t \in \partial \mathcal{D}}$ . Then*

$$\forall z \in \overline{\mathcal{D}}, \quad \forall T \in \mathcal{E}_{n,m} \quad \gamma_z(T) \leq \gamma_{[z]}(T) \leq K(z) \gamma_z(T)$$

with  $K(z) = W^E(z)W^F(z)$ .

PROOF of lemma 3.4. We keep the notation of lemma 3.2 and lemma 3.3. Using a well-known fact, (see [Pi1,Chapter 3] for  $T \in \mathcal{E}_{n,m}$  and  $z \in \overline{\mathcal{D}}$  we may write

$$\begin{aligned} \gamma_z^*(T) &= \gamma_2^*(T : E_z^* \rightarrow F_z) \\ &= \inf \left\{ \pi_z^E(A) \cdot \pi_z^F(B) : T = {}^t B \cdot A, \quad A \in \mathcal{E}_{m,m}, \quad B \in \mathcal{E}_{m,n} \right\}. \end{aligned}$$

By  $(*)$  we obtain

$$\forall z \in \overline{\mathcal{D}}, \quad \forall T \in \mathcal{E}_{n,m} \quad \gamma_z^*(T) \leq K(z) M_z(T) \quad (3.3)$$

with

$$M_z(T) = \inf \left\{ \pi_{[z]}^E(A) \cdot \pi_{[z]}^F(B) : T = {}^t B \cdot A, A \in \mathcal{E}_{m,m}, B \in \mathcal{E}_{m,n} \right\}.$$

Using theorem 2.5 we see easily that  $\{\pi_{[z]}^E\}_{z \in \overline{\mathcal{D}}}$  and  $\{\pi_{[z]}^F\}_{z \in \overline{\mathcal{D}}}$  are subharmonic families of 2-convex norms and by theorem 2.3 we see that  $\{M_z\}_{z \in \overline{\mathcal{D}}}$  is a subharmonic family of norms on  $\mathcal{E}_{n,m}$ ; moreover, for every  $t \in \partial \mathcal{D}$  we have  $M_t = \gamma_t^*$ . Interpolation and duality theorems imply that

$$\forall z \in \overline{\mathcal{D}}, \forall T \in \mathcal{E}_{n,m} \quad M_z(T) \leq (\gamma_{[z]})^*(T). \quad (3.4)$$

From (3.3), (3.4) and by duality we obtain

$$\forall z \in \overline{\mathcal{D}}, \forall T \in \mathcal{E}_{n,m} \quad \gamma_{[z]}(T) \leq K(z) \gamma_z(T) \quad (3.5)$$

On the other hand, we have also

$$\gamma_z(T) = \inf \left\{ \|A : E_z \rightarrow \ell_2^m\| \cdot \|B : F_z \rightarrow \ell_2^m\| : T = {}^t B \cdot A, A \in \mathcal{E}_{m,m}, B \in \mathcal{E}_{m,n} \right\}$$

and the families  $\{\| \cdot : E_z \rightarrow \ell_2^m\| \}_{z \in \overline{\mathcal{D}}}$  and  $\{\| \cdot : F_z \rightarrow \ell_2^m\| \}_{z \in \overline{\mathcal{D}}}$  are clearly subharmonic families of 2-convex norms. Hence by theorem 2.3 we obtain that  $\{\gamma_z\}_{z \in \overline{\mathcal{D}}}$  is a subharmonic family, and the interpolation theorem implies :

$$\forall z \in \overline{\mathcal{D}}, \forall T \in \mathcal{E}_{n,m} \quad \gamma_z(T) \leq \gamma_{[z]}(T) \quad (3.6)$$

This completes the proof of the lemma.

PROOF of theorem 3.1. We assume that we are in one of the cases considered in the theorem, and we keep the notation of the preceding lemmas. By the interpolation theorem  $\{\| \cdot \|_z\}_{z \in \overline{\mathcal{D}}}$  is a subharmonic family of norms, hence

$$\forall z \in \overline{\mathcal{D}}, \forall T \in \mathcal{E}_{n,m} \quad \|T\|_z \leq \|T\|_{[z]}.$$

On the other hand, for every  $t \in \partial \mathcal{D}$  we have clearly

$$\|I : (\mathcal{E}_{n,m}, \gamma_t) \rightarrow (\mathcal{E}_{n,m}, \| \cdot \|_t)\| \leq 1.$$

So by interpolation, we conclude that

$$\forall z \in \overline{\mathcal{D}} \quad \|I : (\mathcal{E}_{n,m}, \gamma_{[z]}) \rightarrow (\mathcal{E}_{n,m}, \| \cdot \|_{[z]})\| \leq 1.$$

Since lemma 3.4 applies in the cases considered in theorem 3.1, we have

$$\forall z \in \overline{\mathcal{D}}, \forall T \in \mathcal{E}_{n,m} \quad \|T\|_{[z]} \leq \gamma_{[z]}(T) \leq K(z) \gamma_z(T) \quad (3.7)$$

Using Kwapien's theorem [Kw] or [Pi1, Chapter 3], in the first case of theorem 3.1 we have

$$\gamma_z(T) \leq \tilde{T}_2(E_z)\tilde{T}_2(F_z)\|T\|_z \quad (3.8)$$

since the Gaussian cotype 2 constant of  $F_z^*$  is smaller than  $\tilde{T}_2(F_z)$ .

By the same theorem, in the second case of theorem 3.1 we have

$$\gamma_z(T) \leq \sqrt{2} \tilde{T}_2(E_z)M^{(2)}(F_z)\|T\|_z \quad (3.9)$$

because the Gaussian cotype 2 constant of  $F_z^*$  is smaller than  $\sqrt{2}M^{(2)}(F_z)$ .

Here we used the simple fact that when we interpolate a family of 2-convex Banach lattices  $\{F_t\}_{t \in \partial\mathcal{D}}$ , we obtain a family of 2-convex Banach lattices  $\{F_z\}_{z \in \overline{\mathcal{D}}}$ . Moreover, the function  $z \mapsto \text{Log } M^{(2)}(F_z)$  is subharmonic. See [He] for more details.

Finally, using theorem 4.1 from [Pi1], instead of Kwapien's theorem, we have in the third case :

$$\gamma_z(T) \leq \left(4M^{(2)}(E_z)M^{(2)}(F_z)\right)^{3/2} \|T\|_z \quad (3.10)$$

If  $\{X_z\}_{z \in \overline{\mathcal{D}}}$  is an interpolation family of finite-dimensional spaces (resp. Banach lattices), then the function  $z \mapsto \text{Log } \tilde{T}_2(X_z)$  (resp.  $z \mapsto \text{Log } M^{(2)}(X_z)$ ) is easily seen to be subharmonic. Using this fact, the estimates (3.8), (3.9), (3.10) and the estimates of  $K(z)$  obtained in lemma 3.2 and lemma 3.3, we obtain from (3.7) that

$$\forall z \in \overline{\mathcal{D}}, \forall T \in \mathcal{E}_{n,m} \quad \|T\|_{[z]} \leq c\|T\|_z.$$

where  $c$  is majorized exactly as in the theorem. This achieves the proof of theorem 3.1.

**Corollary 3.5.** *Let  $E_0, E_1, F_0$  and  $F_1$  be finite-dimensional Banach spaces with  $\dim E_0 = \dim E_1 = m$ ,  $\dim F_0 = \dim F_1 = n$ , and  $\theta \in ]0, 1[$ . Then for all  $T \in \mathcal{E}_{n,m}$*

$$\|T\|_{\mathcal{L}(E_\theta, F_\theta^*)} \leq \|T\|_{[\mathcal{L}(E_0, F_0^*), \mathcal{L}(E_1, F_1^*)]_\theta} \leq c\|T\|_{\mathcal{L}(E_\theta, F_\theta^*)},$$

with  $E_\theta = [E_0, E_1]_\theta$ ,  $F_\theta = [F_0, F_1]_\theta$  and  $c$  is majorized as follows.

1.  $c \leq \left(\max\{\tilde{T}_2(E_i), \tilde{T}_2(F_i) : i = 0, 1\}\right)^2$ .
2.  $c \leq 2\sqrt{\frac{2}{\pi}} \left(\max\{\tilde{T}_2(E_i), M^{(2)}(F_i) : i = 0, 1\}\right)^2$  if the canonical basis of  $\mathbb{C}^n$  is a 1-unconditional basis for  $F_i$  for  $i = 0, 1$ .
3.  $c \leq \frac{32}{\pi} \left(\max\{M^{(2)}(E_i), M^{(2)}(F_i) : i = 0, 1\}\right)^{5/2}$  if the canonical basis of  $\mathbb{C}^m$  (resp.  $\mathbb{C}^n$ ) is a 1-unconditional basis for  $E_i$  (resp.  $F_i$ ) for  $i = 0, 1$ .

**PROOF:** This corollary is a simple consequence of theorem 3.1. We only need to use a simple approximation argument, and theorem 5.1 from [CCRSW2] which makes the link between interpolation families and the complex interpolation method of Calderón.

**Corollary 3.6.** *Let  $E_0, E_1, F_0$  and  $F_1$  be finite-dimensional Banach spaces with  $\dim E_0 = \dim E_1 = m$ ,  $\dim F_0 = \dim F_1 = n$ , and  $\theta \in ]0, 1[$ . Then for all  $T \in \mathcal{E}_{n,m}$*

$$C^{-1}(F) \pi_2(T : F_\theta^* \rightarrow E_\theta) \leq \|T\|_{[\Pi_2(F_0^*, E_0), \Pi_2(F_1^*, E_1)]_\theta} \leq C(E^*) \pi_2(T : F_\theta^* \rightarrow E_\theta)$$

with  $E_\theta = [E_0, E_1]_\theta$ ,  $F_\theta = [F_0, F_1]_\theta$ ; if  $X$  denotes either  $E^*$  or  $F$ ,  $C(X)$  is majorized as follows:

1.  $C(X) \leq \left( \max (\tilde{T}_2(X_0), \tilde{T}_2(X_1)) \right)^2$ .
2.  $C(X) \leq 2\sqrt{\frac{2}{\pi}} \left( \max (M^{(2)}(X_0), M^{(2)}(X_1)) \right)^2$  if the canonical basis of  $\mathbb{C}^p$  ( $p = \dim X_i, i = 0, 1$ ) is a 1-unconditional basis for  $X_i$  for  $i = 0, 1$ .

PROOF: We reproduce the method of Pisier in [Pi3]. Consider the bilinear operator

$$\Phi_i : \mathcal{L}(\ell_2^r, F_i^*) \times \Pi(F_i^*, E_i) \longrightarrow \ell_2^r(E_i),$$

where  $i = 0, 1$ , defined by  $\Phi_i(T, U) = (U \circ T(e_k))_{k \leq r}$  with  $r$  any positive integer and  $(e_k)_{k \leq r}$  is the canonical basis of  $\ell_2^r$ .

For  $i = 0, 1$  we have by definition  $\|\Phi_i\| \leq 1$ , hence by interpolation we obtain, for all  $T \in \mathcal{E}_{m,r}$  and all  $U \in \mathcal{E}_{n,m}$  that

$$\left( \sum_{k=1}^r \|U \circ T(e_k)\|_{E_\theta}^2 \right)^{1/2} \leq \|T\|_{[\mathcal{L}(\ell_2^r, F_0^*), \mathcal{L}(\ell_2^r, F_1^*)]_\theta} \cdot \|U\|_{[\Pi_2(F_0^*, E_0), \Pi_2(F_1^*, E_1)]_\theta}.$$

Using corollary 3.5 we obtain

$$\forall r \in \mathbb{N}, \quad \forall T \in \mathcal{E}_{m,r}, \quad \forall U \in \mathcal{E}_{n,m}$$

$$\left( \sum_{k=1}^r \|U \circ T(e_k)\|_{E_\theta}^2 \right)^{1/2} \leq C(F) \|T\|_{\mathcal{L}(\ell_2^r, F_\theta^*)} \cdot \|U\|_{[\Pi_2(F_0^*, E_0), \Pi_2(F_1^*, E_1)]_\theta}.$$

This clearly implies the first inequality. The second one follows by duality.

*Remark.* It is shown in the proof of lemma 3.4 that, if  $\{E_z\}_{z \in \overline{\mathcal{D}}}$  is an interpolation family, then  $\{\gamma_2(\cdot : E_z \rightarrow E_z)\}_{z \in \overline{\mathcal{D}}}$  is a subharmonic family of norms on  $\mathcal{E}_{n,n}$  where  $n = \dim E_z$ . Taking the identity operator and noting that  $\gamma_2(I : E_z \rightarrow E_z) = d(E_z, \ell_2^n)$ ; the Banach-Mazur distance between  $E_z$  and  $\ell_2^n$  we infer that the function  $z \mapsto \text{Log } d(E_z, \ell_2^n)$  is subharmonic. In particular, if  $E_0$  and  $E_1$  are two  $n$ -dimensional Banach spaces, the function  $\theta \mapsto d([E_0, E_1]_\theta, \ell_2^n)$  is Log-convex on the interval  $[0, 1]$ .

#### 4. The infinite-dimensional case

Let  $X_0, X_1$  be two Banach spaces. The pair  $(X_0, X_1)$  is called an interpolation pair if both spaces  $X_0$  and  $X_1$  are continuously embedded in a Banach space  $\mathcal{U}$  so that we can define their sum and intersection and we can construct the interpolation space  $X_\theta = [X_0, X_1]_\theta$  by the complex interpolation method of Calderón.

In what follows, all interpolation pairs are assumed to have dense intersections, *i.e.*  $X_0 \cap X_1$  is a dense subspace of  $X_k, k = 0, 1$ .

We will say that  $(X_0, X_1)$  is an interpolation pair of Banach lattices if  $X_0$  and  $X_1$  are Banach lattices which can be embedded as lattice-ideals in some  $L^0(\Omega, \mathcal{A}, \mu)$ , with  $(\Omega, \mathcal{A}, \mu)$  a  $\sigma$ -finite measure space. We assume, moreover, that the set of simple functions

$$S_X = \left\{ \sum_{k=1}^n \alpha_k \mathbb{1}_{\Omega_k} : n \in \mathbb{N}, \alpha_k \in \mathbb{C}, \Omega_k \in \mathcal{A} \text{ and } \mu(\Omega_k) < +\infty \right\}$$

is dense in both  $X_0$  and  $X_1$ .

If  $(X_0, X_1)$  is an interpolation pair, and  $G$  is a finite-dimensional subspace of  $X_0 \cap X_1$ , then  $G_k$  denotes  $G$  considered as a subspace of  $X_k$  with  $k = 0, 1$  whereas  $G_\theta$  denotes  $[G_0, G_1]_\theta$  and not  $G$  considered as a subspace  $X_\theta$ .

The following lemma will be the main tool to obtain infinite-dimensional results from the finite-dimensional ones, obtained in the preceding section.

This lemma is not new. The author thanks Professor A.V. Bukhvalov for indicating to him the book **[KP]** as a reference for more general results and for telling him that such results also appear in Russian literature in the works of Aizenshtein and Brudnyi, (see **[Ai]** and **[AB]**). However, we will include its proof for the convenience of the reader.

**Lemma 4.1.** *Let  $(X_0, X_1)$  be an interpolation,  $X$  a subspace of  $X_0 \cap X_1$  which is dense both in  $X_0$  and  $X_1$ , and  $\theta \in ]0, 1[$ . Then for every  $\varepsilon \in ]0, 1[$  and every finite-dimensional subspace  $G$  of  $X$ , there exists a finite-dimensional subspace  $\tilde{G}$  of  $X$  containing  $G$  and such that*

$$\forall x \in G \quad (1 - \varepsilon) \|x\|_{\tilde{G}_\theta} \leq \|x\|_{X_\theta} \leq \|x\|_{\tilde{G}_\theta}.$$

Moreover, if  $(X_0, X_1)$  is an interpolation pair of Banach lattices embedded in  $L^0(\Omega, \mathcal{A}, \mu)$ , and if  $G$  is spanned by finite sequence of characteristic functions of measurable sets, then  $G$  can also be taken of this form.

PROOF: Note that the second inequality is obvious. Let us prove the first one.

Let  $0 < \delta < \frac{\varepsilon}{2}$  and  $\mathcal{N} = \{x_1, \dots, x_m\}$  be  $\delta$ -net in the unit sphere of  $(G, \|\cdot\|_{X_\theta})$ . (*i.e.* for every  $k \leq m$ , we have  $\|x_k\|_{X_\theta} = 1$ , and for every  $y \in G$  with  $\|y\|_{X_\theta} = 1$  there exists  $j \leq m$  such that  $\|y - x_j\|_{X_\theta} \leq \delta$ ). Clearly such a net exists since  $G$  is finite-dimensional.

As  $\|x_k\|_{X_\theta} = 1$ , for every  $k \in \{1, \dots, m\}$  there exists

$$F_k(z) = \sum_{r=1}^{m_k} \Psi_{r,k}(z) x_{r,k}$$

with  $x_{r,k} \in X$ ,  $\Psi_{r,k} \in A(S)$  ( where  $S$  denotes the stripe  $\{z : 0 < \operatorname{Re} z < 1\}$  ) such that  $F_k(\theta) = x_k$  and for every  $t \in \mathbb{R}$  and every  $j = 0, 1$

$$\|F_k(j + it)\|_{X_j} \leq 1 + \delta.$$

Let  $\tilde{G} = \operatorname{span} \{x_{r,k} : 1 \leq k \leq m, 1 \leq r \leq m_k\}$ . This is clearly a finite-dimensional subspace of  $X$  containing  $G$ .

Consider  $x \in G$  with  $\|x\|_{X_\theta} = 1$ , then one may write  $x = y_0 + \sum_{k=1}^{\infty} \lambda_k y_k$  with  $0 \leq \lambda_k \leq \delta^k$  and  $\forall k, y_k \in \mathcal{N}$ . If  $y_k = x_j$  we put  $H_k = F_j$  and define  $F = H_0 + \sum_{k=1}^{\infty} \lambda_k H_k$ . We see immediately that  $F$  is an analytic  $\tilde{G}$ -valued function on  $S$ , continuous on  $\overline{S}$ , satisfying  $F(\theta) = x$  and

$$\forall t \in \mathbb{R}, \forall j = 0, 1 \quad \|F(j + it)\|_{X_j} \leq \frac{1 + \delta}{1 - \delta}.$$

This implies that

$$\|x\|_{\tilde{G}_\theta} \leq \frac{1 + \delta}{1 - \delta} \leq \frac{1}{1 - \varepsilon}.$$

The assertion concerning Banach lattices is clear since, under our assumptions, the space  $X = S_X$  of simple functions is dense both in  $X_0$  and  $X_1$ .

**Theorem 4.2.** *Let  $(E_0, E_1)$  and  $(F_0, F_1)$  be two interpolation pairs. Assume that one of the following properties holds*

1.  $E_0^*$ ,  $E_1^*$ ,  $F_0^*$  and  $F_1^*$  are type 2 spaces .
2.  $E_0^*$ ,  $E_1^*$  are type 2 spaces and  $F_0$ ,  $F_1$  are 2-concave Banach lattices.
3.  $E_0$ ,  $E_1$ ,  $F_0$  and  $F_1$  are 2-concave Banach lattices.

Then  $(E_0 \overset{\vee}{\otimes} F_0, E_1 \overset{\vee}{\otimes} F_1)$  is an interpolation pair, and for  $0 < \theta < 1$  we have

$$[E_0 \overset{\vee}{\otimes} F_0, E_1 \overset{\vee}{\otimes} F_1]_\theta = [E_0, E_1]_\theta \overset{\vee}{\otimes} [F_0, F_1]_\theta.$$

PROOF: Note first that  $E_j \overset{\vee}{\otimes} F_j$ ,  $j = 0, 1$ , is continuously embedded in  $(E_0 + E_1) \overset{\vee}{\otimes} (F_0 + F_1)$ . This is assertion (5) in paragraph 44.4 of Köthe's book [Kö]. So  $(E_0 \overset{\vee}{\otimes} F_0, E_1 \overset{\vee}{\otimes} F_1)$  is an interpolation pair.

Let us make the following convention of notation. If  $(X_0, X_1)$  is an interpolation pair of Banach lattices, then  $X$  denotes the set of simple functions  $S_X$  defined in the begining of this section which is dense in both  $X_0$  and  $X_1$ . If  $X_0$  and  $X_1$  are not assumed to be Banach lattices then  $X$  is simply  $X_0 \cap X_1$ . With this convention, we easily see that  $E \otimes F$  is a dense subspace of  $E_\theta \overset{\vee}{\otimes} F_\theta$ , for all  $\theta \in [0, 1]$ . In order to prove the result it is sufficient to show that

$$\forall T \in E \otimes F \quad \|T\|_{E_\theta \overset{\vee}{\otimes} F_\theta} \leq \|T\|_{[E_0 \overset{\vee}{\otimes} F_0, E_1 \overset{\vee}{\otimes} F_1]_\theta} \leq c \|T\|_{E_\theta \overset{\vee}{\otimes} F_\theta}. \quad (4.1)$$

If  $E_0$ ,  $E_1$ ,  $F_0$  and  $F_1$  are finite-dimensional then  $\mathcal{L}(E_\theta^*, F_\theta) = E_\theta \overset{\vee}{\otimes} F_\theta$  and the theorem in this case is exactly corollary 3.5.

In order to prove the infinite-dimensional case, we will use lemma 4.1. Full details of the argument will be given only in the first case (*i.e.* when  $E_0^*$ ,  $E_1^*$ ,  $F_0^*$  and  $F_1^*$  are spaces of type 2). The other cases are treated similarly using the second part of lemma 4.1 when necessary. Details are easy and left as an exercise for the reader.

We assume that we are under the hypothesis of the first case.

Let  $T = \sum_{k=1}^m x_k \otimes y_k \in E \otimes F$  and  $\varepsilon > 0$ . Define  $X = \text{span } \{x_k : k \leq m\}$  and  $Y = \text{span } \{y_k : k \leq m\}$ . By lemma 4.1 we can find  $G$  and  $H$  finite-dimensional subspaces of  $E$  and  $F$  respectively, such that  $X \subset G$ ,  $Y \subset H$  and

$$\begin{aligned} \frac{1}{\sqrt{1+\varepsilon}} \|x\|_{G_\theta} &\leq \|x\|_{E_\theta} \leq \|x\|_{G_\theta} \quad \forall x \in X \\ \frac{1}{\sqrt{1+\varepsilon}} \|x\|_{H_\theta} &\leq \|x\|_{F_\theta} \leq \|x\|_{H_\theta} \quad \forall x \in Y \end{aligned} \quad (4.2)$$

Let us consider  $x^* \in (G_\theta)^*$  and  $y^* \in (H_\theta)^*$  of norm one. By the Hahn-Banach theorem and (4.2), we can find extensions  $\tilde{x}^* \in (E_\theta)^*$  and  $\tilde{y}^* \in (F_\theta)^*$  with  $\tilde{x}^*|_X = x^*|_X$  and  $\tilde{y}^*|_Y = y^*|_Y$  such that  $\|\tilde{x}^*\|_{(E_\theta)^*} \leq \sqrt{1+\varepsilon}$ ,  $\|\tilde{y}^*\|_{(F_\theta)^*} \leq \sqrt{1+\varepsilon}$ . Hence

$$\left| \sum_{k=1}^m x^*(x_k) y^*(y_k) \right| \leq \|\tilde{x}^*\|_{(E_\theta)^*} \|\tilde{y}^*\|_{(F_\theta)^*} \|T\|_{E_\theta \overset{\vee}{\otimes} F_\theta} \leq (1+\varepsilon) \|T\|_{E_\theta \overset{\vee}{\otimes} F_\theta}.$$

As  $x^*$  and  $y^*$  are arbitrary elements on the unit spheres of  $G_\theta^*$  and  $H_\theta^*$  respectively

$$\|T\|_{G_\theta \overset{\vee}{\otimes} H_\theta} \leq (1+\varepsilon) \|T\|_{E_\theta \overset{\vee}{\otimes} F_\theta}. \quad (4.3)$$

Note that if  $Z$  is a closed subspace of a type 2 Banach space  $W$ , then the quotient  $W/Z$  is also of type 2, and  $\tilde{T}_2(W/Z) \leq \tilde{T}_2(W)$ . From this we deduce that  $\tilde{T}_2(G_k^*) \leq \tilde{T}_2(E_k^*)$  and  $\tilde{T}_2(H_k^*) \leq \tilde{T}_2(F_k^*)$  for  $k = 0, 1$ .

Using corollary 3.5 we find a constant  $c$  depending only on  $E_0$ ,  $E_1$ ,  $F_0$  and  $F_1$  such that

$$\|T\|_{[G_0 \overset{\vee}{\otimes} H_0, G_1 \overset{\vee}{\otimes} H_1]_\theta} \leq c \|T\|_{G_\theta \overset{\vee}{\otimes} H_\theta}. \quad (4.4)$$

Since, for  $k = 0, 1$ , the canonical embedding  $j : G_k \overset{\vee}{\otimes} H_k \rightarrow E_k \overset{\vee}{\otimes} F_k$  has a norm smaller than one, we obtain by interpolation

$$\|T\|_{[E_0 \overset{\vee}{\otimes} F_0, E_1 \overset{\vee}{\otimes} F_1]_\theta} \leq \|T\|_{[G_0 \overset{\vee}{\otimes} H_0, G_1 \overset{\vee}{\otimes} H_1]_\theta}. \quad (4.5)$$

Since  $\varepsilon$  is arbitrary. Combining (4.3), (4.4) and (4.5) we obtain the second inequality of (4.1).

In order to prove the first inequality in (4.1), consider  $T = \sum_{k=1}^m x_k \otimes y_k \in E \otimes F$  and  $\varepsilon > 0$  as before, and define  $X$  and  $Y$  in the same way. Using lemma 4.1, we can find

a finite-dimensional subspace of  $E \otimes F$  containing  $X \otimes Y$ , which we can assume to be of the form  $G \otimes H$ , such that

$$\|T\|_{[(G \otimes H)_0, (G \otimes H)_1]_\theta} \leq (1 + \varepsilon) \|T\|_{[E_0 \overset{\vee}{\otimes} F_0, E_1 \overset{\vee}{\otimes} F_1]_\theta}. \quad (4.6)$$

But for  $k = 0, 1$ , we have  $(G \otimes H)_k = (G \otimes H, \|\cdot\|_{E_0 \overset{\vee}{\otimes} F_0}) = G_k \overset{\vee}{\otimes} H_k$ . So using the easy part of corollary 3.5, we obtain

$$\|T\|_{G_\theta \overset{\vee}{\otimes} H_\theta} \leq \|T\|_{[(G \otimes H)_0, (G \otimes H)_1]_\theta} \quad (4.7)$$

Finally, we have trivially  $\|T\|_{E_\theta \overset{\vee}{\otimes} F_\theta} \leq \|T\|_{G_\theta \overset{\vee}{\otimes} H_\theta}$ . combining this with (4.7) and (4.6), the first inequality of (4.1) follows. This achieves the proof of the theorem.

In what follows, we will need the following known lemma:

**Lemma 4.3.** *Let  $V$  and  $W$  be finite-dimensional subspaces of  $X$  and  $Y$  respectively. Assume that one of the following conditions holds*

1.  $X$  and  $Y$  are type 2 spaces.
2.  $X$  is a type 2 space and  $W$  is a sublattice of the 2-convex Banach lattice  $Y$ .
3.  $V$  and  $W$  are sublattices of the 2-convex Banach lattices  $X$  and  $Y$  respectively.

*Then, there exists a constant  $c_1$  depending only on  $X$  and  $Y$  such that*

$$\forall T \in V \otimes W \quad \|T\|_{X \widehat{\otimes} Y} \leq \|T\|_{V \widehat{\otimes} W} \leq c_1 \|T\|_{X \widehat{\otimes} Y}.$$

**PROOF:** The first case can be found explicitly in [M]. In the other cases, note that if  $u \in \mathcal{L}(V, W^*)$ , there exist a finite-dimensional Hilbert space  $H$ , and two operators  $A : V \rightarrow H$  and  $B : W \rightarrow H$ , such that  $u = {}^t B.A$  and  $\|A\| = \|B\| \leq c_2(\|u\|)^{1/2}$  with  $c_2$  a constant depending only on type 2 ( or 2-convexity ) constants of  $X$  and  $Y$ . Using Maurey's theorem [M] in the type 2 case, or well-known facts in the 2-convex Banach lattice case (see for instance [Pi4, Proposition 1.1]), we can find extensions  $\tilde{A} : X \rightarrow H$  and  $\tilde{B} : Y \rightarrow H$  of the operators  $A$  and  $B$  respectively, such that  $\|\tilde{A}\| \leq c_3 \|A\|$  and  $\|\tilde{B}\| \leq c_3 \|B\|$  with  $c_3$  depending only on type 2 ( or 2-convexity ) constants of  $X$  and  $Y$ . If  $\tilde{u} = {}^t \tilde{B} \cdot \tilde{A} \in \mathcal{L}(X, Y^*)$ , we have  $\|\tilde{u}\| \leq c_1 \|u\|$  and

$$\forall x \in V \ \forall y \in W \quad \langle \tilde{u}(x), y \rangle = \langle u(x), y \rangle.$$

By duality, this implies the difficult inequality of the lemma.

**Theorem 4.4.** Let  $(E_0, E_1)$  and  $(F_0, F_1)$  be two interpolation pairs. Assume that we are in one of the following cases

1.  $E_0, E_1, F_0$  and  $F_1$  are type 2 spaces .
2.  $E_0, E_1$  are type 2 spaces and  $F_0, F_1$  are 2-convex Banach lattices.
3.  $E_0, E_1, F_0$  and  $F_1$  are 2-convex Banach lattices.

Then  $(E_0 \widehat{\otimes} F_0, E_1 \widehat{\otimes} F_1)$  is an interpolation pair, and for  $0 < \theta < 1$  we have

$$[E_0 \widehat{\otimes} F_0, E_1 \widehat{\otimes} F_1]_\theta = [E_0, E_1]_\theta \widehat{\otimes} [F_0, F_1]_\theta.$$

PROOF: Let  $X$  ( resp.  $Y$  ) be a type 2 Banach space or a 2-convex Banach lattice. Then using results from [Pi1, Chapters 3 and 8], we know that every operator from  $X$  into  $Y^*$  factors through a Hilbert space and consequently, every operator from  $X$  into  $Y^*$  is approximable. This implies that the canonical map  $J: X \widehat{\otimes} Y \rightarrow X \widehat{\otimes} Y$  is one to one.

Using this remark, we see that in each case considered in the theorem, the spaces  $E_0 \widehat{\otimes} F_0$  and  $E_1 \widehat{\otimes} F_1$  embed continuously in  $(E_0 + E_1) \widehat{\otimes} (F_0 + F_1)$ . Hence  $(E_0 \widehat{\otimes} F_0, E_1 \widehat{\otimes} F_1)$  is an interpolation pair.

With the same convention of notation as in the begining of the proof of theorem 4.3, the space  $E \otimes F$  is clearly a dense subspace of  $E_\theta \widehat{\otimes} F_\theta$  for every  $\theta \in [0, 1]$ , and so it is also a dense subspace of  $[E_0 \widehat{\otimes} F_0, E_1 \widehat{\otimes} F_1]_\theta$ .

In order to prove the theorem, it is sufficient to find a constant  $c$  such that

$$\forall T \in E \otimes F \quad c^{-1} \|T\|_{E_\theta \widehat{\otimes} F_\theta} \leq \|T\|_{[E_0 \widehat{\otimes} F_0, E_1 \widehat{\otimes} F_1]_\theta} \leq \|T\|_{E_\theta \widehat{\otimes} F_\theta}. \quad (4.8)$$

We give only a detailed proof of this inequality in the first case. The other cases can be treated similarly using the corresponding parts from lemmas 4.1 and 4.3.

In the sequel, we assume that the hypotheses of the first case are satisfied.

Let  $T = \sum_{k=1}^m x_k \otimes y_k \in E \otimes F$  and  $\varepsilon > 0$ . Define  $X = \text{span } \{x_k : k \leq m\}$  and  $Y = \text{span } \{y_k : k \leq m\}$ . By lemma 4.1, we can find a finite dimensional subspace of  $E \otimes F$  containing  $X \otimes Y$ , which we can assume to be of the form  $G \otimes H$ , such that

$$\|T\|_{[(G \otimes H)_0, (G \otimes H)_1]_\theta} \leq (1 + \varepsilon) \|T\|_{[E_0 \widehat{\otimes} F_0, E_1 \widehat{\otimes} F_1]_\theta}. \quad (4.9)$$

By lemma 4.3, the identity map  $j: (G \otimes H)_k \rightarrow G_k \widehat{\otimes} H_k$ ,  $k = 0, 1$  has a norm smaller than  $c_1$  and by interpolation, we obtain

$$\|T\|_{[G_0 \widehat{\otimes} H_0, G_1 \widehat{\otimes} H_1]_\theta} \leq c_1 \|T\|_{[(G \otimes H)_0, (G \otimes H)_1]_\theta} \quad (4.10)$$

Using corollary 3.5 and noting that  $\mathcal{L}(X, Y^*) = (X \otimes Y)^*$  we have

$$\|T\|_{G_\theta \widehat{\otimes} H_\theta} \leq c \|T\|_{[G_0 \widehat{\otimes} H_0, G_1 \widehat{\otimes} H_1]_\theta} \quad (4.11)$$

Note that the constants  $c$  and  $c_1$  depend only on the type 2 constants of the spaces  $E_0, E_1, F_0$  and  $F_1$ .

Trivially, we also have

$$\|T\|_{E_\theta \widehat{\otimes} F_\theta} \leq \|T\|_{G_\theta \widehat{\otimes} H_\theta} \quad (4.12)$$

Combining (4.9), (4.10), (4.11) and (4.12), we obtain the first inequality of (4.8) which is the difficult one.

On the other hand, given  $\varepsilon > 0$  if  $T \in E \otimes F$ , we can write  $T = \sum_{k=1}^{\infty} \alpha_k x_k \otimes y_k$  with  $\alpha_k > 0$ ,  $\|x_k\|_{E_\theta} = \|y_k\|_{F_\theta} = 1$  and  $\sum_1^{\infty} \alpha_k \leq (1 + \varepsilon) \|T\|_{E_\theta \widehat{\otimes} F_\theta}$ . For every  $k \geq 1$ , we can find  $P_k$  ( resp.  $Q_k$  ) an analytic function on the stripe  $S$ , continuous on  $\overline{S}$  with values in  $E_0 + E_1$  ( resp.  $F_0 + F_1$  ) such that  $P_k(\theta) = x_k$  ( resp.  $Q_k(\theta) = y_k$  ) and satisfying that for  $j = 0, 1$  and for every  $t \in \mathbb{R}$  we have  $\|P_k(j + it)\|_{E_j} \leq 1 + \varepsilon$  ( resp.  $\|Q_k(j + it)\|_{F_j} \leq 1 + \varepsilon$  ).

Let us consider the following function  $U(z) = \sum_1^{\infty} P_k(z) \otimes Q_k(z)$ . This is an analytic  $(E_0 + E_1) \overset{\vee}{\otimes} (F_0 + F_1)$ -valued function on  $S$ , continuous on  $\overline{S}$  such that  $U(\theta) = T$  and satisfying:

$$\|U(j + it)\|_{E_j \widehat{\otimes} F_j} \leq (1 + \varepsilon)^2 \sum_{k=1}^{\infty} \alpha_k \leq (1 + \varepsilon)^3 \|T\|_{E_\theta \widehat{\otimes} F_\theta}$$

for every  $t \in \mathbb{R}$  and  $j = 0, 1$ .

This shows that

$$\|T\|_{[E_0 \widehat{\otimes} F_0, E_1 \widehat{\otimes} F_1]_\theta} \leq (1 + \varepsilon)^3 \|T\|_{E_\theta \widehat{\otimes} F_\theta}$$

and proves the second inequality in (4.8), since  $\varepsilon$  is arbitrary.

In order to give the right generalization of corollary 3.6, we need some additional definitions and notation. If  $\{x_k\}_{k \geq 1}$  is a sequence of elements in a Banach space  $X$  we define

$$N_2(\{x_k\}_{k \geq 1}) = \left( \sum_{k=1}^{\infty} \|x_k\|^2 \right)^{1/2}.$$

$$M_2(\{x_k\}_{k \geq 1}) = \sup \left\{ \left( \sum_{k=1}^{\infty} |\langle \xi, x_k \rangle|^2 \right)^{1/2} : \xi \in X^*, \|\xi\| \leq 1 \right\}.$$

Let  $u \in E \otimes F$ . Following Saphar in [Sa], we define the norm

$$d_2(u) = \inf \{M_2(\{x_k\}_{k \geq 1}).N_2(\{y_k\}_{k \geq 1})\}$$

where the infimum runs over all representations of  $u$  of the form  $u = \sum_1^n x_k \otimes y_k$ . And we denote by  $E \overset{d_2}{\widehat{\otimes}} F$  the completion of  $(E \otimes F, d_2)$ .

If we consider the elements of  $E \otimes F$  as operators from  $F^*$  into  $E$  in the natural way, then by [Sa]  $E \overset{d_2}{\widehat{\otimes}} F$  is the closure of  $E \otimes F$  in  $\Pi_2(F^*, E)$  and

$$\forall u \in E \otimes F \quad d_2(u) = \pi_2(u : F^* \rightarrow E).$$

It is also shown that the dual space of  $E \overset{d_2}{\widehat{\otimes}} F$  is isometrically isomorphic to  $\Pi_2(E, F^*)$ .

Using the preceding notation, we can state the following infinite-dimensional generalization of corollary 3.6.

**Theorem 4.5.** *Let  $(X_0, X_1)$  and  $(Y_0, Y_1)$  be two interpolation pairs, and assume that  $X_0^*, X_1^*, Y_0$  and  $Y_1$  are type 2 spaces. Then for  $\theta \in ]0, 1[$  we have*

$$\left[ X_0 \overset{d_2}{\widehat{\otimes}} Y_0, X_1 \overset{d_2}{\widehat{\otimes}} Y_1 \right]_\theta = [X_0, X_1]_\theta \overset{d_2}{\widehat{\otimes}} [Y_0, Y_1]_\theta.$$

*Remark.* A Banach lattice version of this theorem also holds, we leave the detailed formulation as an exercise for the interested reader.

PROOF:(sketch). Since  $X_k \overset{d_2}{\widehat{\otimes}} Y_k$  can be considered as a closed subspace of  $\Pi_2(Y_k^*, X_k)$  ( for  $k = 0, 1$ ), then clearly both  $X_0 \overset{d_2}{\widehat{\otimes}} Y_0$  and  $X_1 \overset{d_2}{\widehat{\otimes}} Y_1$  embed in  $\mathcal{L}(Y_0^* \cap Y_1^*, X_0 + X_1)$  and  $(X_0 \overset{d_2}{\widehat{\otimes}} Y_0, X_1 \overset{d_2}{\widehat{\otimes}} Y_1)$  is an interpolation pair.

Clearly, Corollary 3.6 can be formulated in the following way: There exists a constant  $c > 0$  ( depending only on the type 2 constants of  $X_0^*, X_1^*, Y_0$  and  $Y_1$  ) such that if  $M$  and  $L$  are finite-dimensional subspaces of  $X_0 \cap X_1$  and  $Y_0 \cap Y_1$  respectively, then for all  $T \in M \otimes L$

$$\frac{1}{c} \|T\|_{M \overset{d_2}{\widehat{\otimes}} L} \leq \|T\|_{[M \overset{d_2}{\widehat{\otimes}} L, M \overset{d_2}{\widehat{\otimes}} L]_\theta} \leq \|T\|_{M \overset{d_2}{\widehat{\otimes}} L_\theta}$$

Now, the proof of the theorem goes on exactly as the proof of theorem 4.4, using the following lemma 4.6 instead of lemma 4.3.

**Lemma 4.6.** *Let  $M$  and  $L$  be finite-dimensional subspaces of  $X$  and  $Y$  respectively, and assume that  $Y$  is a type 2 space. Then, for all  $T \in M \otimes L$*

$$\|T\|_{X \overset{d_2}{\widehat{\otimes}} Y} \leq \|T\|_{M \overset{d_2}{\widehat{\otimes}} L} \leq \tilde{T}_2(Y) \|T\|_{X \overset{d_2}{\widehat{\otimes}} Y}.$$

This follows from the fact that for every  $T \in \Pi_2(M, L^*)$ , there exists  $\tilde{T} \in \Pi_2(X, Y^*)$  such that

$$\pi_2(\tilde{T} : X \rightarrow Y^*) \leq \tilde{T}_2(Y) \pi_2(\tilde{T} : M \rightarrow L^*)$$

and

$$\forall x \in M, \forall y \in L \quad \langle T(x), y \rangle = \langle \tilde{T}(x), y \rangle.$$

This, in turn, follows from Pietch's factorization theorem, and Maurey's extension theorem [M].

## 5. Applications

Let us give some consequences of the preceding results.

In the following corollary  $L^p$  will denote  $L^p([0, 1], dt)$  for  $p \in [1, +\infty[$  and the space  $L_0^\infty([0, 1], dt)$  if  $p = +\infty$  (see [BL, chapter 5] for the notation).

**Corollary 5.1.**

1. If  $p_0, p_1, q_0$  and  $q_1$  are elements of  $[1, 2]$ , and  $\theta \in ]0, 1[$ , then

$$[L^{p_0} \overset{\vee}{\otimes} L^{q_0}, L^{p_1} \overset{\vee}{\otimes} L^{q_1}]_\theta = L^{p_\theta} \overset{\vee}{\otimes} L^{q_\theta}.$$

2. If  $p_0, p_1, q_0$  and  $q_1$  are elements of  $[2, +\infty]$ , and  $\theta \in ]0, 1[$ , then

$$[L^{p_0} \widehat{\otimes} L^{q_0}, L^{p_1} \widehat{\otimes} L^{q_1}]_\theta = L^{p_\theta} \widehat{\otimes} L^{q_\theta}.$$

With  $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ ,  $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ .

Using the results of [Z], one can show that for any  $p \in [1, 2]$  and  $\theta \in ]0, 1[$  we have  $[L^2 \widehat{\otimes} L^2, L^2 \widehat{\otimes} L^1]_\theta \neq L^2 \widehat{\otimes} L^p$ . (The proof is based upon the fact that for every even integer  $q$ , there exists a  $\Lambda(q)$  set which is not a  $\Lambda(q + \varepsilon)$  set for any  $\varepsilon > 0$ ). This shows that our results can not be significantly improved and that the assumptions of theorems 4.2 and 4.4 are essential.

*Remark.* In our framework for the interpolation of Banach lattices we excluded the case of  $C(K)$ . It is not difficult to see that we can take for the spaces  $E_0$  or  $F_0$  a  $C(K)$ -space in theorem 4.4, and take  $C([0, 1])$  instead of  $L_0^\infty([0, 1])$  in corollary 5.1.

We also have a non-commutative version of the preceding corollary.

Let us recall first the following definition of unitary ideals. If  $A$  is a compact operator acting on a Hilbert space, then  $|A|$  denotes the modulus of  $A$ , i.e.  $|A| = \sqrt{A^*A}$ . And  $s(A) = \{s_n(A)\}_{n \geq 1}$  denotes the sequence of singular numbers of  $A$ , i.e.  $s_n(A)$  is the  $n$ th eigenvalue of  $|A|$  (where eigenvalues are counted in non-increasing order, according to their multiplicity). Suppose that  $(E, \|\cdot\|_E)$  is a symmetric Banach sequence space. The corresponding unitary ideal  $C_E$  is the space

$$C_E = \{A \text{ compact} : s(A) \in E\}$$

with the norm  $\|A\|_{C_E} = \|s(A)\|_E$  for  $A \in C_E$ .

In the case  $E = \ell_p$ , for  $p \in [1, +\infty[$  we use the notation  $C_{\ell_p} = C_p$  and  $\|A\|_p = (\sum_1^\infty (s_k(A))^p)^{1/p}$  for  $A \in C_p$ . Finally,  $C_\infty$  denotes the space of all compact operators equipped with the usual operator norm.

Using the results of [T] we know that  $C_p$  is of type 2 if  $p \in [2, +\infty[$ . It is also known that  $[C_{E_0}, C_{E_1}]_\theta = C_{E_\theta}$  if  $E_0, E_1$  are symmetric Banach sequence spaces, and  $E_\theta = [E_0, E_1]_\theta$ .

Exploiting these facts, we obtain the following non-commutative analogue of corollary 5.1.

**Corollary 5.2.**

1. If  $p_0, p_1, q_0$  and  $q_1$  are elements of  $]1, 2]$ , and  $\theta \in ]0, 1[$ , then

$$[C_{p_0} \overset{\vee}{\otimes} C_{q_0}, C_{p_1} \overset{\vee}{\otimes} C_{q_1}]_\theta = C_{p_\theta} \overset{\vee}{\otimes} C_{q_\theta}.$$

2. If  $p_0, p_1, q_0$  and  $q_1$  are elements of  $[2, +\infty[$ , and  $\theta \in ]0, 1[$ , then

$$[C_{p_0} \widehat{\otimes} C_{q_0}, C_{p_1} \widehat{\otimes} C_{q_1}]_\theta = C_{p_\theta} \widehat{\otimes} C_{q_\theta}.$$

With  $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ ,  $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ .

As an application of theorem 4.5, we will show the following result, which yields a non-commutative analogue of the fact that if  $1 < p \leq 2$ , then  $\Pi_2(\ell_p, \ell_p)$  is super-reflexive [Pi3].

**Theorem 5.3.** Let  $E$  be a type 2 symmetric Banach sequence space. Then, there exists an interpolation pair  $(\mathcal{H}, X)$ , where  $\mathcal{H}$  is a Hilbert space, and  $\theta \in ]0, 1[$  such that

$$\Pi_2(C_{E^*}, C_{E^*}) = [\mathcal{H}, X]_\theta.$$

**PROOF:** Since  $E$  has type 2,  $E$  is 2-convex and  $q$ -concave for some  $q \in [2, +\infty[$ . Using theorem 2.2 and remark 2.5 of [Pi5], we can find  $E_0$  a 2-convex symmetric Banach sequence space and  $\theta \in ]0, 1[$  such that  $E = [\ell_2, E_0]_\theta$ . If, moreover, we use the reiteration theorem of interpolation and modify the value of  $\theta \in ]0, 1[$ , if necessary, then we can assume that  $E_0$  is  $K$ -convex, hence that  $E_0$  is of type 2.

It follows that  $C_E = [C_2, C_{E_0}]_\theta$  and we know by the results of [GT] that  $C_{E_0}$  is of type 2. Applying theorem 4.5 to the pairs  $(C_2, C_{E_0}^*)$  and  $(C_2, C_{E_0})$  we obtain

$$C_{E^*} \underset{d_2}{\widehat{\otimes}} C_E = \left[ C_2 \underset{d_2}{\widehat{\otimes}} C_2, C_{E_0^*} \underset{d_2}{\widehat{\otimes}} C_{E_0} \right]_\theta.$$

By duality, we obtain

$$\Pi_2(C_{E^*}, C_{E^*}) = [\Pi_2(C_2, C_2), \Pi_2(C_{E_0^*}, C_{E_0^*})]_\theta.$$

Which implies the result since  $\Pi_2(C_2, C_2)$  is a Hilbert space.

**Corollary 5.4.** Let  $E$  be a type 2 symmetric Banach sequence space. Then  $\Pi_2(C_{E^*}, C_{E^*})$  is super-reflexive. In particular, if  $1 < p \leq 2$  then  $\Pi_2(C_p, C_p)$  is super-reflexive.

Note that, by the results of [L], if  $p > 2$  then  $\Pi_2(C_p, C_p)$  is not even  $B$ -convex.

Argueing exactly as in corollary 3.6 and using theorem 4.2, we obtain the following corollary:

**Corollary 5.5.** *Let  $(E_0, E_1)$  and  $(F_0, F_1)$  be two interpolation pairs, and assume that  $E_0^*, E_1^*$  are type 2 space or that  $E_0, E_1$  are 2-concave Banach lattices. Then, for every  $p_0, p_1 \in [1, 2]$  and  $\theta \in ]0, 1[$  the following holds*

$$[\Pi_{p_0}(E_0, F_0), \Pi_{p_1}(E_0, F_1)]_\theta \subset \Pi_p([E_0, E_1]_\theta, [F_0, F_1]_\theta).$$

With  $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ .

Now we give another application of our results. If  $X$  is a complex Banach space and  $0 < p < +\infty$ , then  $\tilde{H}^p(X)$  denotes the closure in  $L^p(\mathbb{T}; X)$  of the set of analytic  $X$ -valued polynomials:

$$\left\{ \sum_{k=0}^n z^k x_k : n \in \mathbb{N}, z \in \mathbb{T} \text{ and } x_k \in X \right\}.$$

The space  $\tilde{H}^p(X)$  equipped with the norm (quasi-norm if  $p < 1$ ) induced by  $L^p(\mathbb{T}; X)$  is a Banach space ( quasi-Banach space).

Let us formulate the following definition:

**DEFINITION.** An interpolation pair  $(X_0, X_1)$  of complex Banach spaces will be called a Hardy-interpolation pair, if and only if, for  $p_0, p_1 \in [1, +\infty[$  and  $\theta \in ]0, 1[$

$$\tilde{H}^{p_\theta}([X_0, X_1]_\theta) \subset \left[ \tilde{H}^{p_0}(X_0), \tilde{H}^{p_1}(X_1) \right]_\theta,$$

with  $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ .

As it is noted in [BX], the converse inequality always holds true with no assumptions on the spaces, and actually, if  $(X_0, X_1)$  is a Hardy-interpolation pair then for all  $p_0, p_1 \in [1, +\infty[$  and  $\theta \in ]0, 1[$

$$\tilde{H}^{p_\theta}([X_0, X_1]_\theta) = \left[ \tilde{H}^{p_0}(X_0), \tilde{H}^{p_1}(X_1) \right]_\theta,$$

with  $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ .

If  $X$  is a complex banach space, then  $(X, X)$  is a Hardy-interpolation pair. This was noted by G. Pisier, see [BX]. It is also shown in [BX] that if  $X_0$  and  $X_1$  are UMD spaces then  $(X_0, X_1)$  is a Hardy-interpolation pair, whereas  $(L^1(\mathbb{T}), c_0(\mathbb{Z}))$  is not a Hardy-interpolation pair.

We think that it would be interesting to study the class of Hardy-interpolation pairs, and one step in this direction is to study the stability properties of this class.

It is not difficult to see that if  $(X_0, X_1)$  is a Hardy-interpolation pair then, for every  $r_0, r_1 \in [1, +\infty[$  the pair  $(L^{r_0}(X_0), L^{r_1}(X_1))$  is a Hardy-interpolation pair where  $L^r(X) = L^r(\Omega, \mathcal{A}, \mathbb{P}; X)$ .

We also have the following:

**Theorem 5.6.** *Let  $X_0, X_1, Y_0$  and  $Y_1$  be complex Banach spaces of type 2. Assume that  $(X_0, X_1)$  and  $(Y_0, Y_1)$  are two Hardy-interpolation pairs. Then  $(X_0 \widehat{\otimes} Y_0, X_1 \widehat{\otimes} Y_1)$  is also a Hardy-interpolation pair.*

PROOF: Let us consider  $p_0, p_1 \in [1, +\infty[,$  then the pairs  $(\tilde{H}^{2p_0}(X_0), \tilde{H}^{2p_1}(X_1))$  and  $(\tilde{H}^{2p_0}(Y_0), \tilde{H}^{2p_1}(Y_1))$  are two interpolation pairs of type 2 spaces. Using theorem 4.4 and the hypothesis we obtain, for every  $0 < \theta < 1,$

$$\begin{aligned} & \left[ \tilde{H}^{2p_0}(X_0) \widehat{\otimes} \tilde{H}^{2p_0}(Y_0), \tilde{H}^{2p_1}(X_1) \widehat{\otimes} \tilde{H}^{2p_1}(Y_1) \right]_\theta = \\ &= \left[ \tilde{H}^{2p_0}(X_0), \tilde{H}^{2p_1}(X_1) \right]_\theta \widehat{\otimes} \left[ \tilde{H}^{2p_0}(Y_0), \tilde{H}^{2p_1}(Y_1) \right]_\theta \\ &= \tilde{H}^{2p_\theta}(X_\theta) \widehat{\otimes} \tilde{H}^{2p_\theta}(Y_\theta). \end{aligned}$$

Let  $\tilde{Q} : \tilde{H}^{2r}(V) \widehat{\otimes} \tilde{H}^{2r}(W) \rightarrow \tilde{H}^r(V \widehat{\otimes} W)$  be the natural norm one operator considered by Pisier in [Pi2] :

$$\tilde{Q} \left( \sum_{k=1}^n f_k \otimes g_k \right) (z) = \sum_{k=1}^n f_k(z) \otimes g_k(z).$$

By interpolation

$$\tilde{Q} \left( \tilde{H}^{2p_\theta}(X_\theta) \widehat{\otimes} \tilde{H}^{2p_\theta}(Y_\theta) \right) \subset \left[ \tilde{H}^{p_0}(X_0 \widehat{\otimes} Y_0), \tilde{H}^{p_1}(X_1 \widehat{\otimes} Y_1) \right]_\theta.$$

By theorem 3.1 of [Pi2], since  $X_\theta$  and  $Y_\theta$  have type 2, we obtain

$$\tilde{Q} \left( \tilde{H}^{2p_\theta}(X_\theta) \widehat{\otimes} \tilde{H}^{2p_\theta}(Y_\theta) \right) = \tilde{H}^{p_\theta}(X_\theta \widehat{\otimes} Y_\theta).$$

Hence,

$$\tilde{H}^{p_\theta}(X_\theta \widehat{\otimes} Y_\theta) \subset \left[ \tilde{H}^{p_0}(X_0 \widehat{\otimes} Y_0), \tilde{H}^{p_1}(X_1 \widehat{\otimes} Y_1) \right]_\theta.$$

And the result follows by a further use of theorem 4.4.

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